ARCCHIT JAIN
10142
7607442699
IIT KANPUR

→ 6 lectures

PROBABILITY THEORY AND STOCHASTIC PROCESSES - GOPAL BHASAK
ISI KOLKATA

- $\Omega \rightarrow$ SAMPLE SPACE : SET OF ALL OUTCOMES

- $\mathcal{A} \rightarrow$ POWER SET OF Ω → Set of all events
  → ALL POSSIBLE UNIONS OF THE ELEMENTS OF Ω

- $P : \mathcal{A} \longrightarrow \mathbb{R}^+$ $\qquad$ $P \rightarrow$ Set valued function
  ↳ Probability Measure
  P satisfies the properties:
  1) $P(\phi) = 0$; $P(\Omega) = 1$
  2) Monotonicity: For $A \subseteq B$, $P(A) \leq P(B)$
  3) Countable Additivity: For $\{A_n\}_{n\geq 1} \in \mathcal{A}$, $P(\bigcup_n A_n) = \sum_{n\geq 1} P(A_n)$

  whenever $\{A_n's\}$ are disjoint

- $(\Omega, \mathcal{A}, P) \rightarrow$ Probability Space

- For $A, B \in \mathcal{A}$, A & B are said to be independent if
  $P(A \cap B) = P(A)P(B)$.
  (ϕ, A), (Ω, A) are sets of independent events.

  $P(A \cap A^c) = P(\phi) = 0$ $\quad$, $P(A)P(A^c) \neq 0$

- Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

σ - Field

Let Ω be the whole set. A collection of subsets satisfying:

1) $\phi, \Omega \in \mathcal{A}$
2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3) If $\{A_n\}_{n \geq 1} \in \mathcal{A}$, then $\bigcup_n A_n \in \mathcal{A}$ and $\bigcap_n A_n \in \mathcal{A}$

Eg: $\{\phi, \Omega\} \rightarrow$ trivial σ-field
$\{\phi, A, A^c, \Omega\} \rightarrow$ Smallest σ-field containing A
σ-field containing A & B
$\{\phi, A, A^c, \Omega, B, B^c, A \cap B, A \cup B, A \setminus B, B \setminus A, (A \cup B)^c, A^c \cup B^c \ldots\}$
or Power set of $\{A \setminus B, B \setminus A, A \cap B, (A \cup B)^c\} = \sigma(\{A, B\})$

Let $\mathcal{C}$ be a collection of subsets of Ω. Then the σ-field generated by $\mathcal{C}$ means $\bigcap_{\mathcal{A} \supseteq \mathcal{C}} \mathcal{A}$ and denoted by $\sigma(\mathcal{C})$

$\sigma(\mathcal{C})$ = Intersection of all the σ-fields containing $\mathcal{C}$.

Exc: σ-field is either finite or uncountable!!

Properties of σ-fields:

1) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two σ-fields, then $\mathcal{A}_1 \cap \mathcal{A}_2$ is σ-field but $\mathcal{A}_1 \cup \mathcal{A}_2$ need not be.

2) Let $\{\mathcal{A}_n\}$ be the sequence of signa fields s.t. $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each $n \geq 1$. Then $\bigcup_{n \geq 1} \mathcal{A}_n$ need not be a σ-field.

- eg: $\mathcal{A}_n = \sigma\left(\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \ldots, \left(\frac{2^n-1}{2^n}, 1\right] \right)$

$\Omega = (0,1]$

$$\bigcap_n \left(\frac{2^n-1}{2^n}, 1\right] = \{1\} \notin \mathcal{A}_n \quad \forall n \geq 1$$

$\Rightarrow \bigcup_n \mathcal{A}_n$ is not a σ-field $\because \bigcup_{n\geq 1} \mathcal{A}_n \not\ni \{1\}$

otherwise $\{1\} \subseteq \mathcal{A}_n$ for some n

But $\mathcal{A}_n$ contains $\left(\frac{2^n-1}{2^n}, 1\right]$ as an atom $\Rightarrow \{1\}$ can't be contained in $\mathcal{A}_n$.

$\because \{1\} = \bigcap_n \underbrace{\left(\frac{2^n-1}{2^n}, 1\right]}_{B_n} \Rightarrow$ If $\bigcup \mathcal{A}_n$ is a σ-field, then $\{1\}$ should belong to $\bigcup_n \mathcal{A}_n$.

$\because B_k \in \mathcal{A}_k$ for each $k \Rightarrow B_k \subseteq \bigcup_n \mathcal{A}_n \quad \forall k$

$\Rightarrow \bigcap B_k$ should belong to $\bigcup_n \mathcal{A}_n \Rightarrow\Leftarrow$

Another example: $\Omega = \{1, 2, \ldots\}$ and $\mathcal{A}_n = \sigma(\{1\}\{2\} \ldots, \{n\})$

$\bigcup_n \mathcal{A}_n$ do not contain set of odd numbers or even numbers.

- Binomial: $\Omega = \{\{0\}, \ldots, \{n\}\}$

$P(K) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad 0 < p < 1$

- Poisson: $\Omega = \{0, 1, 2, \ldots\}$ $\quad \mathcal{A} = P(\Omega)$

$$P(k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \text{for } \lambda > 0$$

Atomic σ-field : $\{A_1, A_2, \ldots, A_n, \ldots\}$

$\hookrightarrow$ Partitions of Ω

$P(A_1) = a_1 \geq 0 \quad a_2 \geq, \quad \ldots \quad a_n \geq 0 \quad \ldots$

Non Atomic

Borel σ-field : σ-field generated by the open sets.
or σ-field " " " intervals.
(because every open set can be written as a countable union of open intervals)

$\mathcal{A} = \sigma(\{(a, b] : a, b \text{ rationals}\}) \rightarrow$ Atoms are all singletons

$$\bigcap_{n \geq 1} \left(r - \frac{1}{n}, r\right] = \{r\}$$

Irrationals also belong to $\mathcal{A}$ coz we can construct 2 sequences of rationals which sandwich any irrational.

A is an atom if for any $B \in \mathcal{A}$ s.t $B \subseteq A$ then $B = A$ or $B = \phi$

$\mathcal{A}$ generated by singleton in R
$= \{$all countable & co-countable subsets in $R\}$

Examples : 1) $PM((a, b]) = b - a$

2) $PM((a, b]) = \int_a^b f(x)\, dx$ for any $f \geq 0$ and

$\int_a^b f(x)\, dx < \infty$ for any a, b (reals)

For Probability: $P((a,b]) = \int_a^b f(x)\,dx$

where $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

• Ex: $\mathcal{C} = \{A_1, A_2, \ldots\}$ Then $\sigma(\mathcal{C})$ contains an atom (Smallest σ-field containing $\mathcal{C}$)

$\{\bigcap_{i \geq 1} D_i : D_i = A_i$ or $A_i^c\}$ Show that these sets are atoms of $\sigma(\mathcal{C})$ whenever they are non-empty.

<u>Converse</u>: Does every σ-algebra which contains an atom countably generated?

(Perhaps No!)

• Let A be uncountable set and x be an element $\notin A$.
$\sigma(A \cup \{x\})$ has x as an atom but 'may' be uncountably generated.
• $\mathbb{R}$ is uncountable $\Rightarrow$ $P(\mathbb{R})$ is an uncountably generated σ-field
But its atoms are singletons of $\mathbb{R}$.

• Let $\mathcal{C} = \{$<u>finite</u> disjoint union of $(a,b]$, a, b, reals$\}$

$P\left(\bigcup_i (a_i, b_i]\right) = \sum_i \int_{a_i}^{b_i} f(x)\,dx$ → FINITE ADDITIVE MEASURE

• <u>Field</u> $\mathcal{C}$ is a field if:
1) $\Phi, \Omega \in \mathcal{C}$
2) If $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$
3) If $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$

Is $\mathcal{G}$ a σ-field?

Step 1: Need to show P is countably additive on $\mathcal{C}$. FIELD

If $\sum_{i\geq 1} c_i < \infty$ and $c_i \geq 0$ then $\forall\, \epsilon > 0$, $\exists\, N \geq 1$ s.t. $\sum_{i=N+1}^{\infty} c_i < \epsilon$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \geq P\left(\bigcup_{i=1}^{N} A_i\right)$$
$$= \sum_{i=1}^{N} P(A_i)$$ whenever $\{A_i\}$ are pairwise disjoint for all N ($A_i \cap A_j = \phi \;\; \forall\, i, j$)

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} P(A_i)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i \setminus \left(\bigcup_{i=1}^{N} A_i\right)\right) < \epsilon$$

$$\bigcup_{i=1}^{N} A_i \subseteq (-n, n]$$

Step 2: Extend P to monotone limits of $\mathcal{C}$, say, to $\mathcal{G}$.

$$\mathcal{G} = \{A : A = \lim A_n \text{ whenever } A_n \uparrow A \text{ or } A_n \downarrow A \text{ for } \{A_n\} \in \mathcal{C}\}$$

$$P(A) = \lim P(A_n) \qquad \boxed{\mathcal{G} \supset \mathcal{C}}$$

Step 3 $\quad P^*(A) = \inf_{\{A_i\}} \left\{ P\left(\bigcup_{i\geq 1} A_i\right) : \cup A_i \supseteq A \text{ and } \cup A_i \in \mathcal{G} \right\}$

OUTER MEASURE

1) P^* is countably sub-additive

Exc. Show that if $\mathcal{M}$ is a monotone class containing $\mathcal{C}$, then $\mathcal{M} \supseteq \sigma(\mathcal{C})$

Definition: A monotone class $\mathcal{M}$ is a collection of sets s.t. whenever $\{A_n\} \in \mathcal{M}$, $A_n \uparrow A$ or $A_n \downarrow A$, then $A \in \mathcal{M}$

$(A_n \in \underset{\text{BIGGER SET}}{\Omega})$

3) $P^*(A \cup A^c) \overset{?}{=} P^*(A) + P^*(A^c)$

$\mathcal{F} = \{A \subseteq \Omega : P^*(A) + P^*(A^c) = 1\}$

Step 4: Show that $\mathcal{F}$ is a field

Step 5: P^* is finite additive on $\mathcal{F}$.

Step 6 P^* is countably additive on $\mathcal{F}$

Step 7: $\mathcal{F}$ is a σ-field & hence $\mathcal{F} \supseteq \sigma(\mathcal{C})$

Step 8: If $A \in \mathcal{F}$, & $P^*(A) = 0$ then $B \in \mathcal{F}$ for any $B \subseteq A$ & $P^*(B) = 0$

$|\mathcal{B}(\mathbb{R})| = |\mathbb{R}|$ $\qquad |\mathcal{F}| = 2^{|\mathbb{R}|}$

$\boxed{G}$

$\phi \to 0$

$\{0\} \to 1$

$\{0,1\} \to 2$

$\vdots$

$\{0, 1, \ldots\} \to \omega$ ORDINAL NUMBER

$\mathbb{N} \cup \{\omega\} \to \omega + 1$

First Uncountable Ordinals $= \alpha$

$$\boxed{\sigma(\mathcal{C}) \overset{?}{=} \bigcup_{\beta < \alpha} \mathcal{C}_\beta}$$

where

$\mathcal{C} = \{(a,b] : a, b \text{ rationals}\}$

$\mathcal{C}_1 = \{$countable unions & compliments of ~~∞~~ sets of $\mathcal{C}\}$

$\mathcal{C}_2 = \{$ " " " " " " $\mathcal{C}_1\}$

$P(A) = \int_A 1 \, dx$ where $A \in \mathcal{F}$ $\hookrightarrow$ Event Space $A \subseteq \Omega$

$\int_\Omega \cancel{Q} \, 1_A \, dx = \int_A dP = P(A)$

$\int_\Omega \sum_i^{n} c_i I_{A_i} \, dP = \sum_{i=1}^{n} c_i P(A_i)$ $\rightarrow$ FINITE!!

. f satisfying $f^{-1}(-\infty, x] \in \mathcal{F}' \ \forall x$

$\sup \{ \int s \, dP, \ 0 \leq s \leq f \} = \int f \, dP$
& $s \rightarrow$ SIMPLE

$P(A) = \int_A f(x) \, dx$ $A \in \mathcal{F}$,

$\hookrightarrow = \int_A dP$ (just a notation)

Eg: $f = 1$ on $(0,1) = \Omega$

$\int_{(0,1)} dx = 1$ $\rightarrow$ $\int_A d\lambda$

dx $\downarrow$ MEASURE

$d\lambda$ $\downarrow$ LEBESGUE MEASURE

A $\hookrightarrow$ ANY MEASURABLE SET.

- Stochastic Processes & Applications : Bhattacharya & Waymire
- Real Analysis & Probability by ROBERT R B Ash

$$P(A) = \int_A dP = \int_\Omega I_A \, dP$$

let $f = \underbrace{\sum_{i=1}^{n} c_i I_{A_i}}_{\text{Simple function (Takes finite values)}} = \begin{cases} c_i & \forall x \in A_i \end{cases}$ A_i's are partitions of Ω. (W.L.O.G.) $A_i \in \mathcal{F}$

then, $\int_\Omega f \, dP = \int_\Omega \sum c_i I_{A_i} \, dP$

$$= \sum_{i=1}^{n} c_i \int_\Omega I_{A_i} \, dP$$

$$= \sum_{i=1}^{n} c_i \int_{A_i} dP$$

$$= \sum_{i=1}^{n} c_i P(A_i)$$

Measurable Functions

$f : \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{F}, P)$ is a Probability Space

If $f^{-1}(B) \in \mathcal{F}$ for any $B \in \underbrace{\mathcal{B}(\mathbb{R})}_{\text{Borel Set}}$

↳ σ-field generated by the open intervals

then such f is called a Measurable Function.

Consider a collection $\mathcal{C} = \{ B \in \mathcal{B}(\mathbb{R}) \mid f^{-1}(B) \in \mathcal{F} \}$

Let's check if $\mathcal{C}$ is a σ-field.

$f^{-1}(\phi) = \phi \in \mathcal{F} \Rightarrow \phi \in \mathcal{C}$

$f^{-1}(\mathbb{R}) = \Omega \in \mathcal{F} \Rightarrow \mathbb{R} \in \mathcal{C}$

let $A \in \mathcal{C}$ $\Rightarrow f^{-1}(A) \in \mathcal{F}$
$\Rightarrow (f^{-1}(A))^c \in \mathcal{F}$
$\Rightarrow f^{-1}(A^c) \in \mathcal{F}$
$\Rightarrow A^c \in \mathcal{C}$

$f^{-1}(A))^c = f^{-1}(A^c)$ (Not True in general)

in general $f(A \cap B) \neq f(A) \cap f(B)$

$f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$

$\Rightarrow$ If f is measurable, $\mathcal{C} = \{ B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{F} \}$

• f is measurable if $f^{-1}(-\infty, x] \in \mathcal{F}$ for all $x \in \mathbb{R}$

$\Rightarrow \{ y \in \Omega \mid f(y) \leq x \} \in \mathcal{F}$ for all $x \in \mathbb{R}$

$\mathcal{C} \supseteq \{ (-\infty, x] \mid x \in \mathbb{R} \}$

$(-\infty, a] \cap (-\infty, c]^c = (c, a]$

c a

let f be a measurable function, $f \geq 0$

Define $\int_\Omega f \, dP = \sup_{s} \left\{ \int_\Omega s \, dP : 0 \leq s \leq f \right\}$

$s \to$ simple functions

Check linearity: $\int_\Omega (f+g) \, dP = \int_\Omega f \, dP + \int_\Omega g \, dP$ wherever they are defined.

$\int_\Omega c f \, dP = c \int_\Omega f \, dP$ $c \in \mathbb{R}$

I Let $f \geq 0$ be a measurable function and bounded

Define $f_n = \begin{cases} \dfrac{k-1}{2^n} & \dfrac{k-1}{2^n} \leq f(x) < \dfrac{K}{2^n} \\ n & f(x) \geq \dfrac{n\,2^n}{2^n} \end{cases}$

$$\int f_n \, dP = \sum_{k=1}^{n2^n} \left[\frac{k-1}{2^n} P\left(f^{-1}\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) + n P\left(f^{-1}\left([n, \infty)\right)\right) \right]$$

Note : 1) $f_n \uparrow$ in n

2) $f - f_n \leq \frac{1}{2^n} \longrightarrow 0$ as $n \to \infty$

for large enough $n \geq \sup_x |f(x)|$

II Let $f \geq 0$ measurable, need not be bounded.

$f_M = f \wedge M = \text{Min}(f, M)$ $\qquad M \geq 0$

for each x,

$|f_M - f| \to 0$ as $M \to \infty$ (Pointwise Convergence)

$\int f_M \, dP \to \int f \, dP$

Monotone Convergence Theorem (MCT):

Let $f_n \geq 0$ be measurable functions s.t $f_n \uparrow f$. then f is a measurable function.

Eg: Let $f_n \geq 0$ be measurable

1) $f_n \uparrow f \Rightarrow f$ is measurable

2) $\limsup f_n = \inf_{k \geq 1} \sup_{n \geq k} f_n$ is also measurable.

3) $\lim f_n$ (when exists) is also measurable.

$$\int f \, dP = \int \lim_{n \to \infty} f_n \, dP = \lim_{n \to \infty} \int f_n \, dP$$

III For f measurable: Take $f = f^+ - f^-$

where $f^+ = \text{Max}(f, 0)$, $f^- = \text{Max}(-f, 0)$

$\int f \, dP = \int f^+ dP - \int f^- dP$ whenever at least one of the integral in RHS is finite

Fatou's lemma: If $f_n \geq 0$ are measurable, then

$$\liminf_n \int_\Omega f_n \, dP \geq \int_\Omega (\liminf_n f_n) \, dP$$

$$\liminf = \sup_{k \geq 1} \inf_{n \geq k} f_n$$

let $g_k = \inf_{n \geq k} f_n$. Use MCT on g_k

$$\Rightarrow g_k \leq f_n \quad \forall n \geq k$$

$$\Rightarrow \int g_k \, dP \leq \int f_n \, dP \quad \forall n \geq k \quad \text{(Monotonicity of integral)} \longrightarrow \circledast$$

$\because \int f \, dP = \sup_s \left(\int s \, dP \ \ \forall s: s \leq f \right)$

$$\lim_{k \to \infty} \int g_k \, dP = \int_\Omega \lim_{k \to \infty} g_k \, dP$$

$$= \int_\Omega (\liminf f_n) \, dP$$

Thus from $\circledast$ $\int g_k \, dP \leq \inf_{n \geq k} \int f_n \, dP$

$$\Rightarrow \lim_{k \to \infty} \int g_k \, dP \leq \lim_{k \to \infty} \inf_{n \geq k} \int f_n \, dP$$

$$\Rightarrow \int (\liminf f_n) \, dP \leq \liminf \int f_n \, dP$$

Lebesgue
Dominated Convergence Theorem

If $|f_n| \leq g$ & $f_n \to f$ all measurable, and g is integrable,
$\Rightarrow g^+ \;\&\; g^- < \infty \quad \forall x$

i.e. $\int |g| \, dP < \infty$

Then 1) $\int f_n \, dP \to \int f \, dP$

(Use $g - f_n \geq 0$
& $g + f_n \geq 0$)

2) $\int_\Omega |f_n - f| \, dP \to 0$

eg $f_n = n I_{(0, \frac{1}{n}]}$ $\qquad f_n \to 0$

$\int f_n \, dP = n P\left((0, \frac{1}{n}]\right) = n \frac{1}{n} = 1 \quad \forall n$

↓

P is uniform.

EXPECTATIONS $\qquad (\Omega, \mathcal{F}, P)$ Probability Space

Defn: A random Variable $X : \Omega \to \mathbb{R}$

$E(X^r) = \int_\Omega X^r \, dP$ whenever they exists.

μ is σ-finite means $\exists \{A_n\}$ DISJOINT Partition of Ω $\left(\Omega = \bigcup_n A_n\right)$ s.t.
$\mu(A_n) < \infty \;\forall n$

$X : \Omega \to \mathbb{R}$

Define
$(P \circ X^{-1})(A) = P(X^{-1}(A))$

induced measure on $\mathcal{B}(\mathbb{R})$

$$(P \circ X^{-1})((-\infty, x]) = P(X^{-1}((-\infty, x]))$$
$$= P(\omega : X(\omega) \leq x)$$
$$= P(X \leq x)$$
$$= F(x) > 0$$

$\hookrightarrow$ Distribution function of X.

Properties of F:

1) F is right continuous $(F(x) = F(x^+))$ $\forall x$
2) F is non decreasing
3) $F(x)$ is 1 as $x \to \infty$
   $F(x) \to 0$ as $x \to -\infty$

$$\mu_P((a, b]) = (P \circ X^{-1})((a, b]) = F(b) - F(a)$$

$$\int_\Omega X(\omega)\, dP = \int_{\mathbb{R}} x \, d(P \circ X^{-1})((-\infty, x]) = \int x \, dF(x)$$

Bounded Variation: (BV)

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have BV property if

$$\underset{\{t_i\}}{\operatorname{Max}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$$

$a = t_0 < t_1 < t_2 \ldots < t_n = b$

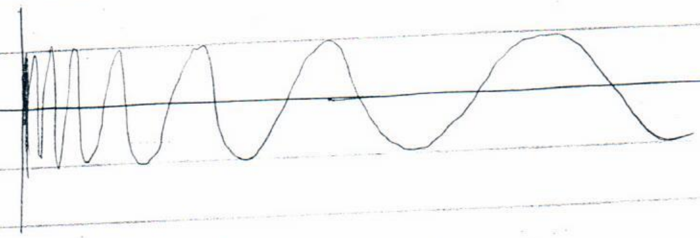
- Any BV function can be written as a difference of 2 non decreasing function.
- Any non decreasing function is a BV function.

Absolutely Pointwise Uniform

Continuous Integrable Differentiable Convergent

Sequence Series Function

Not all bounded functions are BV: $\boxed{f(x) = \sin(\frac{1}{x})}$



$f(x)$ is not BV near 0

<u>Absolutely continuous functions</u> $f: \mathbb{R} \to \mathbb{R}$ on $[a,b]$

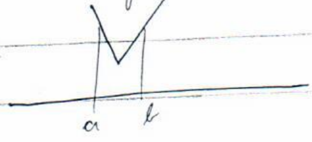
$\forall \epsilon > 0, \exists \delta > 0$ s.t. $\sum_{j=1}^{n} (t_j - t_{j-1}) < \delta$

$\Rightarrow \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| < \epsilon$ where $a \leq t_0 < t_1 < \ldots < t_n \leq b$

Uniform continuous:

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, y$ s.t. $|x - y| < \delta$,
$|f(x) - f(y)| < \epsilon$

Ex: Check if it's absolutely continuous (AC)

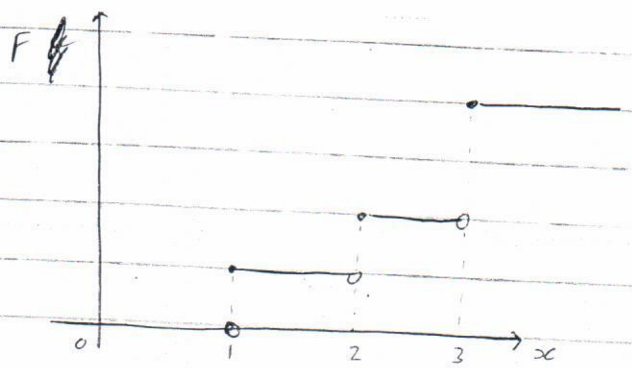


- $AC \Rightarrow BV$
- AC functions $\Rightarrow f'$ exists almost everywhere (a.e.)

- $\exists$ a continuous function which is not BV ($f(x) = \sin(1/x)$)
- $\exists$ a BV function which is not continuous.

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Check if f is BV, AC in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- 

$$F(x) = \begin{cases} 1 & \text{for } x \geq 3 \\ 1/2 & \text{for } 2 \leq x < 3 \\ 1/4 & \text{for } 1 \leq x < 2 \\ 0 & \text{for } x < 1 \end{cases}$$

$$P\left(\left(\tfrac{1}{2}, \tfrac{3}{2}\right]\right) = \int_{1/2}^{3/2} dF(x) = P\left(\left(\left(\tfrac{1}{2}, 1\right) \cup \left(1, \tfrac{3}{2}\right]\right) \cup \{1\}\right)$$

$$= \underbrace{\int_{\frac{1}{2}}^{1^-} dF(x)}_{0} + \underbrace{\int_{1}^{3/2} dF(x)}_{0} + \underbrace{\int_{1}^{1} dF(x)}_{\text{II}} = F(1^+) - F(1^-) = \frac{1}{4}$$

$$\int_{1/2}^{3/2} x\, dF(x) = \int_{(1/2, 1)} x \cdot 0\, dx + \int_{\{1\}} x\, dF(x) + \int_{(1, 3/2)} x \cdot 0\, dx = 1\left(F(1) - F(1^-)\right)$$

$$= 1 \cdot (1/4 - 0) = \frac{1}{4}$$

- $E(X) = \int X\, dP = \int x\, dF$

$$E\left(E(Y|X)\right) = E(Y)$$

CONDITIONAL EXPECTATION $(\Omega, \mathcal{F}, P)$

let $\exists$ a σ-field $\mathcal{G} \subseteq \mathcal{F}$ $\qquad \underbrace{\{\phi, \Omega\}}_{\text{TRIVIALLY}} \subseteq \mathcal{F}$

$$\min_{f \text{ is } \mathcal{G} \text{ measurable}} E(X - f)^2$$

~~$E(f) = E(X)$~~

Then $\operatorname{argmin} f = E(X|\mathcal{G})$

where $\int_A E(X|\mathcal{G})\, dP = \int_A X\, dP \quad \forall A \in \mathcal{G}$

$$E\left(X - E(X|\mathcal{G})\right)^2 = E\left(X - f + f - E(X|\mathcal{G})\right)^2$$

$$= E(X - f)^2 + E\left(f - E(X|\mathcal{G})\right)^2 + 2E(X - f)E\left(f - E(X|\mathcal{G})\right)$$

$$E\left[(X - f)(f - E(X|\mathcal{G}))\right] = E\left(E\left((X - f)(f - E(X|\mathcal{G}))\,|\,\mathcal{G}\right)\right)$$

$$= E\left((f - E(X|\mathcal{G}))\underbrace{E((X - f)|\mathcal{G})}_{E(X|\mathcal{G}) - f}\right) = -E\left(f - E(X|\mathcal{G})\right)^2$$

$$\Rightarrow E\left(X - E(X|\mathcal{G})\right)^2 = E(X - f)^2 - E\left(f - E(X|\mathcal{G})\right)^2 \leq E(X - f)$$

and equality iff $f = E(X|\mathcal{G})$

Useful Properties:

1) If f is measurable w.r.t $\mathcal{G}$ then $E(f|\mathcal{G}) = f$ almost surely.

2) If f is $\mathcal{G}$ measurable & h $\mathcal{F}$-measurable then $E(fh|\mathcal{G}) = fE(h|\mathcal{G})$

3) $E(E(h|\mathcal{G})) = E(h)$

Examples: Let $\mathcal{G} = \{\phi, \Omega\}$ $\quad E(X|\mathcal{G}) = E(X)$ a.s. in $\mathcal{G}$

Only Measurable functions of $\mathcal{G}$ are constants $f \equiv c$

- If $\mathcal{G} = \{\phi, B, B^c, \Omega\}$

then $f = \begin{cases} c_1 & \text{on } B \\ c_2 & \text{on } B^c \end{cases}$ is a $\mathcal{G}$ measurable function.

- Let $X = I_A$ (Indicator Function)

$$\int_B E(X|\mathcal{G})\, dP = \int_B X\, dP = \int_B I_A\, dP = \int I_A I_B\, dP = \int I_{A\cap B}\, dP = P(A\cap B)$$

$$\downarrow$$

$$\int_B (c_1 I_B + c_2 I_{B^c})\, dP = c_1 P(B\cap B) + c_2 P(B\cap B^c) = P(A\cap B)$$

$$\Rightarrow c_1 = \frac{P(A\cap B)}{P(B)} = P(A|B)$$

$$\int_{B^c} E(X|\mathcal{G})\, dP = \int_{B^c} I_A\, dP = P(A\cap B^c) \quad \Rightarrow c_2 = P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)}$$

$$\Rightarrow E(X|\mathcal{G}) = \begin{cases} P(A|B) & \text{for } \omega \in B \\ P(A|B^c) & \text{for } \omega \in B^c \end{cases}$$

$$\int_A f\,dP = \int_B g\,dP \quad \forall A \in \mathcal{G}$$

$\Rightarrow \underbrace{f = g}_{\mathcal{G}\text{ measurable}}$ a.s. w.r.t $\left[P|_{\mathcal{G}}\right]$ ↳ measure

Eg: $g(x) = x$; $f(x) = \begin{cases} x & \text{on Irrationals} \\ 0 & \text{on Rationals} \end{cases}$

$\Rightarrow$ $f = g$ almost surely ~~on~~ w.r.t $[\lambda]$ ↳ Lebesgue Measure

If $\mathcal{G} = \sigma(\{B_n\})$ where $\{B_n\}$ are partitions of Ω.

Then $f(\omega) = C_n$ when $\omega \in B_n$ are the only $\mathcal{G}$ measurable functions.

$E(X|\mathcal{G}) = P(A|B_n)$ if $\omega \in B_n$ almost surely w.r.t. $\left[P|_{\mathcal{G}}\right]$

If $X = \sum c_i I_{A_i}$ then $E(X|\mathcal{G}) = (\sum a_i E(I_{A_i}|\mathcal{G}))$

↳ SIMPLE FUNCTION

$$= \sum_{i=1}^{n} a_i B_n \text{ if } \omega \in B_n \text{ a.s.}$$

$$P(A_i|B_n) \overset{\text{NOTATION}}{=} E(I_{A_i}|B_n) = \frac{\int I_{A_i \cap B_n}\,dP}{P(B_n)} = \frac{\int_{B_n} I_{A_i}\,dP}{P(B_n)}$$

$$E(X|B_n) = \sum a_i \frac{\int_{B_n} I_{A_i}\,dP}{P(B_n)} = \frac{\int_{B_n} X\,dP}{P(B_n)} \quad \text{when } \omega \in B_n \text{ a.s.}$$

For non-neg. X take over simple function

For general X, write $X = X^+ - X^-$

MARTINGALES

Let $\{X_n\}_{n\geq 0}$ be a seq. of Random Variable on $(\Omega, \mathcal{F}, P)$ satisfying
1) $E(|X_n|) < \infty \quad \forall n$
2) $E(X_{n+1} | X_1, X_2, \ldots, X_n) = X_n$

Consider $\Omega = \{HH, HT, TH, TT\}$
let X = No of heads $\quad X = 0, 1, 2$

$\underbrace{X^{-1}\{0\} = \{TT\}}_{B_1}$; $X^{-1}\{1\} = \underbrace{\{HT, TH\}}_{B_2}$; $X^{-1}\{2\} = \underbrace{\{HH\}}_{B_3}$

let $Y = \begin{cases} 1 & \text{if } 1^{st} \text{ toss is head} \\ 0 & \text{otherwise} \end{cases}$ $\quad Y \to$ simple function.

$E(Y|X) = E\left(\mathbb{1}_{\{HH, HT\}} \mid \sigma(X)\right)$ $\quad \sigma(X) \to$ σ-field generated by X
smallest σ-field containing $\{TT\}, \{TH, HT\}, \{HH\}$

$$= \begin{cases} P(\{HH, HT\} \mid X^{-1}\{0\}) & \text{if } \omega \in X^{-1}\{0\} = \{TT\} \\ P(\{HH, HT\} \mid X^{-1}\{1\}) & \text{if } \omega \in X^{-1}\{1\} = \{HT, TH\} \\ P(\{HH, HT\} \mid X^{-1}\{2\}) & \text{if } \omega \in X^{-1}\{2\} = \{HH\} \end{cases}$$

$$= \begin{cases} 0 \\ \dfrac{P(\{HT\})}{P(\{HT, TH\})} \\ \dfrac{P(\{HH\})}{P(\{HH\})} = 1 \end{cases}$$

σ - field generated by 2 Random Variables :

$\sigma(X_1, X_2) = \sigma\left(\{X_1^{-1}(B_1) \cap X_2^{-1}(B_2)\} \text{ where } B_1, B_2 \in \mathcal{B}(\mathbb{R})\right)$

$\underbrace{\sigma(X_1)}_{\mathcal{F}_1} \subseteq \underbrace{\sigma(X_1, X_2)}_{\mathcal{F}_2} \subseteq \underbrace{\sigma(X_1, X_2, X_3)}_{\mathcal{F}_3} \ldots \subseteq \sigma(X_1, \ldots, X_n)$

↳ NATURAL FILTERATION

Esc ① Show that $S_n = X_0 + X_1 + \ldots + X_n$ where $P(X_i = 1) = p$, $P(X_i = -1) = q = 1 - p$.

Then $\{S_n\}$ is a Martingale w.r.t. $\mathcal{F}_n$ (filterations) $= \sigma(\{X_0, \ldots X_n\})$ when $p = q = \frac{1}{2}$

② $W_n = S_n^2 - n$ → compensated Martingale

③ $Z_n = \left(\frac{q}{p}\right)^{S_n}$ is an exponential Martingale w.r.t. $\{\mathcal{F}_n\}$

④ $Y_n = S_n - n(p - q)$ is a Martingale

⑤ $H_n = \exp\left\{S_n - \frac{n\sigma^2}{2}\right\}$ where $X_i \sim N(0, \sigma^2)$

=

Conditional expectation $E(X|\mathcal{G})$ is the best estimator (in mean square error sense) of X given the information set $\mathcal{G}$.

Greater the information $\mathcal{G}$, better the estimation of X.

A sequence $\{X_n\}_{n \geq 1}$ is a Martingale w.r.t a filteration $\{\mathcal{F}_n\}$ if it satisfies
1) $E|X_n| < \infty \quad \forall n$
2) $E(X_{n+1} | \mathcal{F}_n) = X_n$

· Filteration: $\mathcal{F}_n$ is a sequence of increasing σ-field and X_n is measurable w.r.t $\mathcal{F}_n$

Esc 1 Soln: For $p = q = \frac{1}{2}$, To show $S_n = X_0 + X_1 + \dots + X_n$ is a Martingal where $P(X_i = 1) = p$, $P(X_i = -1) = q$ & X_i's are iid (independent and identically distributed) random variables.

1) $E|S_n| \leq \underbrace{E|X_0|}_{c} + \underbrace{E|X_1|}_{1} + \dots + \underbrace{E|X_n|}_{1}$

$= c + n < \infty$ $\qquad |X_i| = 1 \quad \forall i \geq 1$

2) $E(S_{n+1} | S_0, S_1, \dots, S_n)$

$= E(S_{n+1} | X_0, X_1, \dots, X_n)$

$= E(S_n + X_{n+1} | \mathcal{F}_n)$

$= E(S_n | \mathcal{F}_n) + E(X_{n+1} | \mathcal{F}_n)$

$= S_n + E(X_{n+1})$

$= S_n + 0$ $\qquad X_{n+1}$ is independent of all other X_i

$= S_n$

Esc 3 Soln: $Z_n = \left(\frac{q}{p}\right)^{S_n}$

1) $E|Z_n| = E\left(\left(\frac{q}{p}\right)^{S_n}\right)$

$= E\left(\left(\frac{q}{p}\right)^{X_0 + X_1 + \dots X_n}\right)$

$= \left(\frac{q}{p}\right)^c E\left(\frac{q}{p}\right)^{X_1} E\left(\frac{q}{p}\right)^{X_2} \dots E\left(\frac{q}{p}\right)^{X_n}$

$= \left(\frac{q}{p}\right)^c \left(E\left[\left(\frac{q}{p}\right)^{X_1}\right]\right)^n$

$= \left(\frac{q}{p}\right)^c < \infty$

$E\left(\frac{q}{p}\right)^{X_1} = \left(\frac{q}{p}\right)^{-1} q + \left(\frac{q}{p}\right)^1 p$

$= p + q = 1$

Gambler's Ruin Problem:

START $\downarrow$ 0 — 20 — 100

P(Win) = p

P(Loss) = $q = 1-p$

Probability of achieving 100 before going Bankrupt??

2)
$$E(Z_{n+1} \mid Z_0, Z_1, \ldots, Z_n)$$
$$= E(Z_{n+1} \mid S_0, S_1, \ldots, S_n)$$
$$= E(Z_{n+1} \mid X_0, X_1, \ldots, X_n)$$
$$= E(Z_{n+1} \mid \mathcal{F}_n)$$
$$= E\left(\left(\frac{q}{p}\right)^{S_{n+1}} \Big| \mathcal{F}_n\right)$$
$$= E\left(\overbrace{\left(\frac{q}{p}\right)^{S_n}}^{\text{KNOWN}} \left(\frac{q}{p}\right)^{X_{n+1}} \Big| \mathcal{F}_n\right)$$
$$= \left(\frac{q}{p}\right)^{S_n} E\left(\left(\frac{q}{p}\right)^{X_{n+1}} \Big| \mathcal{F}_n\right)$$
$$= \underbrace{\left(\frac{q}{p}\right)^{S_n}}_{Z_n} \underbrace{E\left(\left(\frac{q}{p}\right)^{X_{n+1}}\right)}_{1}$$
$$= Z_n$$

Gambler's Ruin Problem

a — c — b

$\tau \to$ Stopping Time $\quad \tau : \Omega \to \{0, 1, \ldots, \infty\}$ satisfying

$$\{\tau \le n\} \in \mathcal{F}_n = \sigma(X_1, \ldots, X_n)$$

$X_i \to$ Outcome of i^{th} game $= \pm 1$ (Win/Loss)

$\boxed{P_c(\tau_{100} < \tau_0)} = ??$

$c = 20$ (starting Point)

$\phi(c) = \boxed{P_c(\tau_b < \tau_a)}$

$c \xrightarrow{p} c+1$, $\quad c \xrightarrow{q} c-1$

Recurrence relation (with boundary conditions):

$$\phi(c) = p\,\phi(c+1) + q\,\phi(c-1) \qquad \phi(a) = 0, \ \phi(b) = 1$$

$$p\,[\phi(c+1) - \phi(c)] = q\,[\phi(c) - \phi(c-1)]$$

$$\boxed{\tau_y = \inf\{n \ge 0 \mid S_n = y\}}$$

Doob Optional Stopping Theorem (DOST):
Let $\{Y_n\}$ be a martingale w.r.t $\{\mathcal{F}_n\}$ & τ be a stopping time w.r.t $\{\mathcal{F}_n\}$.

Then $E(Y_\tau) = E(Y_0)$ or $E(Y_\tau | \mathcal{F}_0) = Y_0$
provided 1) $P(\tau < \infty) = 1$

2) $E|Y_\tau| < \infty$

3) $E(Y_m | \tau > m) \to 0$ as $m \to \infty$

$\because E(Y_{n+1}) = E(E(Y_{n+1} | \mathcal{F}_n)) = E(Y_n) = \dots = E(Y_0)$.

$E(E(Y_{n+1}|\mathcal{F}_n)\mathcal{F}_{n-1}) = E(Y_n | \mathcal{F}_{n-1})$

$\hookrightarrow E(Y_{n+1} | \mathcal{F}_{n-1}) = Y_{n-1}$

$E(Y_{n+1} | \mathcal{F}_0) = Y_0$

Show that the hypothesis of DOST holds for Gambler's Ruin Problem
i.e. $E(Y_\tau | \mathcal{F}_0) = Y_0$

$p = q = \frac{1}{2}$ | $Y_n = S_n$ $\quad \tau = \tau_a \wedge \tau_b$

$c = E_c(S_\tau)$

$= a \underbrace{P_c(\tau_a < \tau_b)}_{1 - P_c(\tau_b < \tau_a)} + b P_c(\tau_b < \tau_a)$

$c = (b-a) \overbrace{P_c(\tau_b < \tau_a)}^{\phi(c)} + a$

$\Rightarrow \boxed{\phi(c) = \frac{c-a}{b-a}}$

If $Y_n = Z_n = \left(\frac{q}{p}\right)^{S_n}$ $\Rightarrow E_c(Z_\tau) = \left(\frac{q}{p}\right)^c = \left(\frac{q}{p}\right)^a P_c(\tau_a < \tau_b) + \left(\frac{q}{p}\right)^b P_c(\tau_b < \tau_a)$

If $E(Y_{n+1} | \mathcal{F}_n) \geq Y_n \quad \forall n$ then $\{Y_n\} \rightarrow$ Submartingale
If $E(Y_{n+1} | \mathcal{F}_n) \leq Y_n \; \forall n$ then $\{Y_n\}$ is a 'Supermartingale'

Properties: 1) If $\{Y_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and ϕ is a convex function then $\{\phi(Y_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ is properly defined provided $E|\phi(Y_n)| < \infty \; \forall n$.

Jensen's Inequality: $E(\phi(x)) \geq \phi(E(x))$

Similarly, concave functions $\longrightarrow$ Supermartingale

Properties: 1) ~~If~~ Let $\{Y_n\}$ be a martingale w.r.t $\{\mathcal{F}_n\}$ & $\{A_n\}$ be a predictable sequence and integrable sequence i.e A_n is measurable w.r.t $\mathcal{F}_{n-1}$.

Define $X_n = Y_n - Y_{n-1}$ Martingale difference sequence
Then $Z_n = \sum_{k=1}^{n} A_k X_k$ is a Martingale w.r.t. $\mathcal{F}_n$.

2) Doob-Meyer Decomposition: Let $\{H_n\}$ be a submartingale w.r.t $\{\mathcal{F}_n\}$ Then $\exists \{A_n\}$, an increasing predictable integrable sequence s.t. $H_n - A_n = M_n$ is a martingale w.r.t. $\{\mathcal{F}_n\}$ i.e.
$H_n = A_n + M_n$ and the decomposition is unique with $A_0 = 0$.

eg: $\underbrace{M_n}_{\text{Martingale}} = \overbrace{S_n^2}^{\text{SUBMARTINGALE}} - \underbrace{n}_{A_n \text{ with } A_0 = 0}$

4) Suppose Y is a random variable & $E(|Y|) < \infty$ & $\{\mathcal{F}_n\}$ is a filteration. Then $X_n = E(Y|\mathcal{F}_n)$ is a martingale w.r.t $\{\mathcal{F}_n\}$

$$A_n = \sum_{k=1}^{n} \left(E(H_k | \mathcal{F}_{k-1}) - H_{k-1} \right) \geq 0$$

$$M_n = H_n - A_n = (H_n - H_{n-1}) - (A_n - A_{n-1}) + \underbrace{(H_{n-1} - A_{n-1})}_{M_{n-1}}$$

$$E(M_n | \mathcal{F}_{n-1}) = \underbrace{E(H_n | \mathcal{F}_{n-1}) - H_{n-1}}_{A_n - A_{n-1}} - (A_n - A_{n-1}) + M_{n-1}$$

$$= M_{n-1}$$

$\Rightarrow M_n \longrightarrow$ Martingale

Submartingale Convergence Theorem

Let $\{Y_n\}$ be a submartingale w.r.t $\{\mathcal{F}_n\}$ & $\sup_n E(|Y_n|) < \infty$

Then $\exists\, Y^*$ (s.t $E(|Y^*|) < \infty$) s.t

$$Y_n \to Y^* \quad \text{as } n \to \infty \text{ almost surely}$$

- If $E(Y_n) \to E(Y_\infty)$ then $E|Y_n - Y_\infty| \to 0$
- If $Z_n = \left(\frac{q}{p}\right)^{S_n}$ $\quad E(Z_n) = \left(\frac{q}{p}\right)^{c} < \infty$

$\Rightarrow \exists\, Z_\infty$ s.t $Z_n \longrightarrow Z_\infty$

Claim $\boxed{Z_n \longrightarrow 0}$

(Use Strong Law of Large Numbers) ...

Markov Processes: "The future, given present, is independent of the past

$\{X_n\} \to$ [Discrete state space, (or) continuous " "] with [Discrete time, or Continuous time]

Spatial Processes : $X_{(t,y)}$ eg $\to$ Location of a species ; Not directional for discrete state space

A Stochastic Process $\{X_n\}_{n \geq 0}$ is said to be a Markov Chain or Markov Process if

Transition Probability

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

$\forall (x_0, x_1, \ldots, x_n, x_{n+1}), \ \forall n$

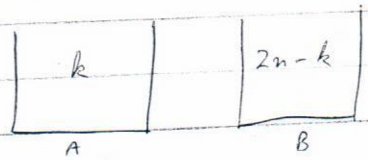
whenever $P(X_0 = x_0, \ldots, X_n = x_n) > 0$

- A transition is homogenous if $P(X_{n+1} = y \mid X_n = x) = P(X_1 = y \mid X_0 = x) \ \forall n \geq 1$

- example Let S be the state space of MC & it is finite.

EHRENFEST MODEL

$S = \{0, \ldots, 2n-1\}$



Let X_n = No of Molecules in A at time n, 2N molecules

$$P(X_{n+1} = y \mid X_n = x) = \begin{cases} \frac{2N - x}{2N} & y = x+1 \\ \frac{x}{2N} & y = x - 1 \\ 0 & \text{otherwise} \end{cases}$$

This process is 'irreducible' as there is always a +ve probability of getting shuffled. $i \sim j \Rightarrow \exists n$ s.t. $P(X_n = j \mid X_0 = i) > 0$

Equillibrium distribution: π

$$\pi(k) = \binom{2N}{k} \frac{1}{2^{2N}} \rightarrow \text{BINOMIAL} \qquad k = 0, \ldots, 2N$$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & \cdots & 2N \\ \hline 0 & 0 & 1 & 0 & & \\ 1 & \frac{1}{2N} & 0 & 1-\frac{1}{2N} & & \\ 2 & & & & & \\ \vdots & & & \ddots & & \\ \vdots & & 0 & & \ddots & \\ 2N & & & & 1 & 0 \end{array}_{(2N+1)\times(2N+1)}$$

one step transition matrix

$P^k \to$ k-step transition matrix

$$\underbrace{P^k(i,j)}_{(i,j)^{TH} \text{ element of } P^K} = P(X_k = j \mid X_0 = i)$$

$$\frac{1}{n+1}\sum_{k=0}^{n} P^k \longrightarrow \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \underbrace{\pi}_{\text{equilibrium distribution}} \end{bmatrix} \quad \text{as } n \to \infty$$

$$P(X_1 = j) = \sum_{i=0}^{2N} P(X_1 = j, X_0 = i) = \sum_i P(X_0 = i)\, P(X_1 = j \mid X_0 = i) = \sum_i (\pi_i\, p_{ij}) = \pi_j$$

$\pi \to$ Stationary / invariant distribution

<u>Theorem</u>: Let $\{X_n\}$ be an irreducible Markov Chain on a finite state space with period 1.

- Period of i = $d(i)$ = gcd $\{n : P_{ii}^n > 0\}$
- If $i \sim j$, then $d(i) = d(j)$.
  - $\Rightarrow$ For an irreducible M.C., there is one period, say d.

Esc: A Random Walk $\{S_n\}$ is a M.C. with period 2.
Is there a stationary distribution?
No there is no Probability dist, but counting is invariant.

Esc: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad 0 \rightleftarrows 1 \quad$ $P^2 = I$

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k = \frac{I + P + I + P + \cdots}{n} = \frac{I+P}{2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \Rightarrow \boxed{\pi = (1/2, 1/2)}$$

$$\Rightarrow \boxed{\pi P = \pi}$$

$(i,j)^{th}$ element of k-step transition matrix

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) = E(I_{\{X_n = j\}} \mid X_0 = i)$$

$$\Rightarrow \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{k} = \frac{1}{n+1} \sum_{k=0}^{n} E(I_{\{X_k = j\}} \mid X_0 = i) = \frac{1}{n+1} E\left(\sum_{k=0}^{n} I_{\{X_k = j\}} \mid X_0 = i\right)$$

$$= E\left(\frac{1}{n+1}\left(\quad \right) \mid X_0 = i\right)$$

Then $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \longrightarrow \pi_j$ as $n \to \infty$

If $\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty$ iff $p = q = \frac{1}{2}$ $\Rightarrow \{0\}$ is recurrent state

If $p \neq q \Rightarrow \sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$ $\Rightarrow \{0\}$ is Transient state

- For an irreducible M.C., all states have similar properties.

- Recurrent
  - Null Recurrent if $E_0(\tau_0) = \infty$ even if $P_0(\tau_0 < \infty) = 1$
  - Positive Recurrent if $E_i(\tau_i) < \infty$

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \approx \frac{1}{\sqrt{\pi n}}$$ (using Sterling's approximation for $n!$)

- For Gambler's Ruin Problem: a and b are the absorbing states

$\phi(c) = P(\underbrace{X_n \text{ will reach } b \text{ before } a}_{A} \mid X_0 = c)$

$$= \sum_{j=a}^{b} P(A, X_1 = j \mid X_0 = c)$$

$$= \sum_j P(A \mid X_1 = j, X_0 = c) \times P(X_1 = j \mid X_0 = c)$$

$$= \sum_j P(A \mid X_1 = j)\, p_{cj} = \sum_j P(A \mid X_0 = j)\, p_{cj} = \sum_j \phi(j)\, p_{cj}$$

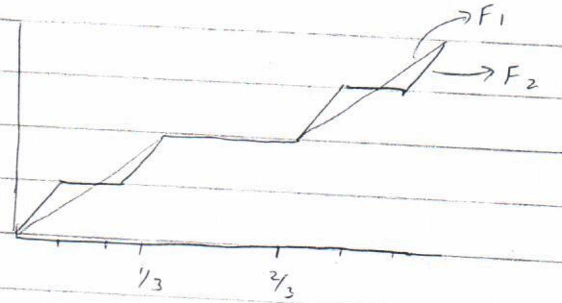
$$\phi(c) = p\,\phi(c+1) + q\,\phi(c-1)$$

- A Markov Chain is irreducible if for any $x, y \in S$ $\exists n$ s.t $p^{(n)}(x, y) > 0$

where $p^{(2)}(x, y) = \int_S p(x,z)\,p(z,y)\,dz$ (Density)

- $P(X_n \in U_y \mid X_0 = x) > 0$ for all neighbourhood of U_y s.t $\lambda(U_y) > 0$
  $\uparrow$ LEBESGUE MEASURE

- $P(X_n \in U_y \mid X_0 \in U_x) = \dfrac{P((X_n \in U_y) \cap (X_0 \in U_x))}{P(X_0 \in \quad)}$



F_1
F_2
$\vdots$
$F_n \downarrow F$
$\hookrightarrow$ continuous but nowhere absolutely continuous

Consider $\{X_t\}$ Continuous time Stochastic Process.

Markov Property: $P(X_t \in U \mid X_{t_1} \in U_{t_1}, \ldots, X_{t_k} \in U_{t_k})$

$= P(X_t \in U \mid X_{t_k} \in U_{t_k})$ whenever $P(X_{t_i} \in U_{t_i}) > 0$ $(\forall i = 1, \ldots, k)$

$\{\mathcal{F}_s\}_{s>0}$ $\quad \mathcal{F}_s \subset \mathcal{F}_v$ for $s < v$

$E(f(X_t) \mid \mathcal{F}_s) = E(f(X_t) \mid \sigma(X_s))$ $\quad s < t$

X_s ~~are~~ is measurable w.r.t $\mathcal{F}_s$

· For $\mu = 0, \sigma = 1$, BM is called Standard Brownian Motion Process.

$$\mathcal{F}_s = \sigma(X_u \quad 0 \le u_i < s)$$
$$= \sigma\left(X_{u_1}^{-1}(B_1) \cap \dots \cap X_{u_K}^{-1}(B_K)\right) \qquad B_k \in \mathcal{B}(\mathbb{R}), \ \forall k \ge 1$$
$$0 \le u_1 \le u_2 < \dots < u_k \le s$$

· $\sigma(X_0, X_1, \dots) = \sigma\left(\bigcup_{n=1} \sigma(X_0, X_1, \dots, X_n)\right)$ or $\sigma\left(\bigcup_{n=1} \sigma(X_n)\right)$

$$\mathcal{B}(\mathbb{R}^{\mathbb{N}}) = \sigma\{\underbrace{B_1 \times B_2 \times \dots \times B_K \times R \times R \dots}_{\text{Finite Dimensional Rectangles}} \mid k \ge 1\}$$

$B_i \to$ OPEN INTERVALS, or
" sets, or
Borel sets

· Finite disjoint union of such rectangles form a field

· $\mathcal{B}(\mathbb{R}^{[0,\infty)}) = \bigcup_{\{t_i\}} \sigma(R^{\{t_i\}})$

$\{t_i\} \to$ sequence in $[0,\infty)$

· $\{B_t\}_{t \ge 0}$ is a Brownian Motion process if

1) $t \to B_t$ is continuous (CONTINUOUS SAMPLE PATH)
$B_0 = x$ (or $B_0 = 0$) with P_x probability 1.

2) $B_t - B_s \mid \mathcal{F}_s$ $(0 \le s < t)$ is independent of $\mathcal{F}_s$ (INDEPENDENT INCREMENT)

3) $B_t - B_s \sim N(\mu(t-s), \sigma^2(t-s))$

4) $(B_{t_1}, \dots, B_{t_K}) \sim$ Multivariate Normal (μ, Σ) where $\Sigma_{(i,j)} = \text{Min}(t_i, t_j)$

· $N(t) \to$ No of arrivals in queue upto time t $\to$ Point Process.

$N(t) - N(s)$ is independent of $\mathcal{F}_s$

· $N(t)$ is Poisson Process if $P(N(t) - N(s) = k \mid \mathcal{F}_s) = e^{-\lambda(t-s)} \dfrac{(\lambda(t-s))^K}{k!}$